Period 3, Sept 9, 2024

a) If $f(x) = \frac{1}{x}$, what is $f(c)$? $f(c+h)$?

$F(c) = \frac{1}{c}$

$F(c+h) = \frac{1}{c+h}$

b) If $f(x) = x^3 + 3x$, what is $f(c)$? $f(c+h)$?

$F(c) = c^3 + 3c$

$F(c+h) = (c+h)^3 + 3(c+h) = (c^3 + 3c^2h + 3ch^2 + h^3) + 3(c+h)$

c) If $f(x) = \sqrt{2x-2}$, what is $f(c)$? $f(c+\Delta x)$?

$F(c) = \sqrt{2c-2}$

$F(c+\Delta x) = \sqrt{2(c+\Delta x) - 2}$

d) How do you find the Slope between two points?

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{F(x_2) - F(x_1)}{x_2 - x_1}$

$\lim_{h \to 0} \frac{F(x+h) - F(x)}{(x+h) - x}$

Slope of 2 Points That are Really close

$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$

$\frac{d}{dx}(F(x)) = F'(x) = \frac{dy}{dx} = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$

Example 1: ~~Find the slope of the graph of $f(x) = 2x - 3$ at the point (2, 1)~~

$c =$ ______

IRC aT $x = c$

Tangent Slope aT $x = c$ $= \lim_{x \to c} \frac{F(x) - F(c)}{x - c}$

Slope of Tangent Line aT $x = c$

IRC aT $x = c$

$F'(x) = \lim_{h \to c} \frac{F(x+h) - F(x)}{h} = m$ of Tangent Line

$F'(c)$

Average Rate of Change v.s. Instantaneous Rate of Change (IRC)

$$Average\ Rate\ of\ Change = \frac{f(b) - f(a)}{b - a}$$

DEFINITION: Instantaneous Rate of Change

The **instantaneous rate of change** of *f* at *c* is the limit as *x* approaches *c* of the average rate of change. Symbolically, the instantaneous rate of change of *f* at *c* is

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Alternate Form of Instantaneous ROC

The **Instantaneous ROC** of *f* at a real number *c* has been defined as the real number $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$.

The alternative form of the **Instantaneous ROC** of f at a real number c is

$$f'(c) = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$$

or

$$f'(c) = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x}$$

Example 5:

(a) Find the int. rate of change of $f(x) = \sqrt{2x}$ at $x = 8$.

$c =$ ______

Line $\Rightarrow$ $y = mx+b$

$m =$ slope (x, y) Point

Find Tangent and Normal lines as well.

$$F'(8) = \frac{1}{\sqrt{2\cdot 8}} = \frac{1}{\sqrt{16}} = \frac{1}{4}$$

Point

$$F(8) = \sqrt{2\cdot 8} = \sqrt{16} = 4$$

$(8, 4)$ $m = \frac{1}{4}$

$y = mx+b$

$4 = \frac{1}{4}(8)+b$

$4 = 2+b$

$2 = b$

$y = \frac{1}{4}x+2$

$F(x+h) = \sqrt{2(x+h)}$ slope at X

$$\lim_{h\to 0} \frac{\left(\sqrt{2(x+h)} - \sqrt{2x}\right)}{h} \cdot \frac{\left(\sqrt{2(x+h)} + \sqrt{2x}\right)}{\left(\sqrt{2(x+h)} + \sqrt{2x}\right)}$$

$$\lim_{h\to 0} \frac{2(x+h) - 2x}{h\left(\sqrt{2(x+h)} + \sqrt{2x}\right)}$$

$$\lim_{h\to 0} \frac{\cancel{2x} + 2h - \cancel{2x}}{h\left(\sqrt{2(x+h)} + \sqrt{2x}\right)} = \lim_{h\to 0} \frac{2\cancel{h}}{\cancel{h}\left(\sqrt{2(x+h)} + \sqrt{2x}\right)}$$

$$\frac{2}{\sqrt{2x} + \sqrt{2x}} = \frac{\cancel{2}}{\cancel{2}\sqrt{2x}} = \frac{1}{\sqrt{2x}} = F'(x)$$

$F'(x) = \frac{dy}{dx} = \frac{\Delta y}{\Delta x} = \frac{d}{dx}\left(F(x)\right) =$ equation For The Slope of Tangent Line

normal Line is perpendicular To Tangent Line

Tangent Slope is $\frac{a}{b}$, Normal Slope $-\frac{b}{a}$

Normal Line at $X = 8$

$m = -\frac{4}{1} = -4$

Point $(8, 4)$

$y = -4x+b$

$4 = -4(8)+b$

$4 = -32+b$

$36 = b$

Slope at 8

$$\lim_{x\to 8} \frac{F(x) - F(8)}{X - 8}$$

$$\lim_{x\to 8} \frac{\left(\sqrt{2x} - 4\right)}{(X-8)} \cdot \frac{\left(\sqrt{2x}+4\right)}{\left(\sqrt{2x}+4\right)}$$

$$\lim_{x\to 8} \frac{2x - 16}{(X-8)\left(\sqrt{2x}+4\right)}$$

$$\lim_{x\to 8} \frac{2\cancel{(x-8)}}{\cancel{(x-8)}\left(\sqrt{2x}+4\right)}$$

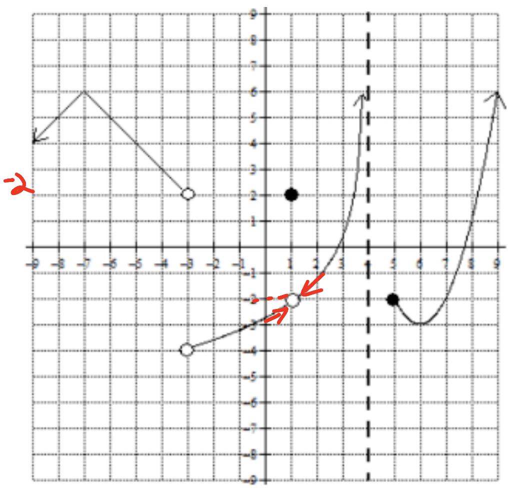
$$\frac{2}{\sqrt{2\cdot 8}+4} = \frac{2}{4+4} = \frac{2}{8}$$

$m = \frac{1}{4}$

$\sqrt{16} = 4$

does not exist, give a reason.

1.	$\lim_{x\to -3^+} f(x) + \lim_{x\to 5^+} 3f(x)$	
2.	$\lim_{x\to 1}\left[\frac{1}{2}f(x)+\cos(\pi x)\right]$	$\frac{1}{2}(-2)+-1=-1+-1=-2$
3.	$\lim_{x\to 4^-} f(x)$	
4.	$\lim_{x\to -\infty} f(x)$	
5.	$\lim_{x\to -3} f(x)$	



$\lim_{x\to 1} F(x) = -2$

$\lim_{x\to 1} \cos \pi x = \cos \pi = -1$

8. $\lim_{x\to 3^+} \frac{x^2-4}{x^2-x-6} = \lim_{x\to 3^+} \frac{(x-2)\cancel{(x+2)}}{(x-3)\cancel{(x+2)}}$

$\lim_{x\to 3^+} \frac{(x-2)}{x-3} = \frac{1}{+RSN} = +\infty$

$= DNE$

9. $\lim_{x\to 2^-} \ln(-x+2) = -\infty$

$= DNE$

$\ln x = y$

$e^y = x$

$-\infty < y < \infty$

$x > 0$

$2^- \Rightarrow 1.999$

$\ln(-1.999+2)$

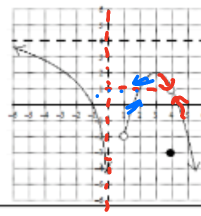
$\ln(.001) \approx -7$

$\ln(.00000001) \approx -18$

For question 12 – 16, use the equation $g(x)$ below and the graph of the function $f(x)$.

$$g(x) = \begin{cases} 3|x+3|, & x < -2 \\ \cos\left(\frac{\pi x}{2}\right), & -2 \le x < 2 \\ ax^2 + 2x, & x \ge 2 \end{cases}$$

Graph of $f(x)$



Cont inious at c

$\lim_{x\to c} F(x)$ Exists

$F(c)$ Exists

$\lim_{x\to c} F(x) = F(c)$

12. Is $g(x)$ is ~~continuous~~ NOT continuous at $x = -2$. [Base your response on the three part definition of continuity.]

$\lim_{x\to -2} g(x) = g(-2)$

$\lim_{x\to -2^+} \cos\left(\frac{\pi x}{2}\right) = -1$

$\lim_{x\to -2^-} 3|x+3| = 3$

NOT The Same

$\lim_{x\to -2} g(x) = DNE$

13. For what value(s) of a is $g(x)$ continuous at $x = 2$?

$\lim_{x\to 2^+} g(x) = \lim_{x\to 2^-} g(x)$

$a(2)^2 + 2(2) = \cos \frac{2 \cdot \pi}{2}$

$4a + 4 = \cos \pi$

$4a + 4 = -1$

$-4 \quad -4$

$4a = -5$

$a = -\frac{5}{4}$

14. For what value(s) of b is the function $f(x)$ discontinuous? At which of these values does $\lim_{x\to b} f(x)$ exist? Explain your reasoning.

$x = 0$

$[0, 1]$

$x = 4$

$\lim_{x\to 0^-} F(x) = -\infty = DNE$

$\lim_{x\to 1^-} F(x) = DNE$

$\lim_{x\to 4} = 1$

15. Find $\lim_{x\to 2^+} [g(x) + 2f(x)]$. $= -1 + 2(1) = -1 + 2 = 1$

$\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+} -\frac{5}{4}(x^2) + 2x = -\frac{5}{4}(2)^2 + 2 \cdot 2 = -\cancel{\frac{5}{4} \cdot 4} + 4 = -1$

$\lim_{x\to 2} F(x) = 1$

For #15, use your answer from #13.

16. Which of the following limits do(es) not exist? Give a reason for your answers.

$\lim_{x\to 1} f(x)$	$\lim_{x\to 4} f(x)$	$\lim_{x\to 0^-} f(x)$
$\lim_{x\to 1^-} F(x) = DNE$	1	$\lim_{x\to 0^-} F(x) = -\infty$ DNE

17. Find the values of k and m so that the function below is continuous on the interval $(-\infty,\infty)$.

$$f(x)=\begin{cases} x^2-kx+3, & x<-2 \\ 2x-3, & -2\le x\le 3 \\ 3-2m, & x>3 \end{cases}$$

$3-2m = 2(3)-3$

Find m

$\lim_{x\to -2^-}(x^2-kx+3) = (-2)^2-k(-2)+3 = 4+2k+3 = 7+2k$

$\lim_{x\to -2^+}(2x-3) = 2\cdot -2-3 = -4-3 = -7$

$\lim_{x\to -2} F(x) \Rightarrow \cancel{7}+2k = -7 \Rightarrow 2k = -14$
$\quad\quad -7 \quad\quad -7$

$k=-7$

Rich People Rule

25. $\lim_{x\to-\infty}\dfrac{2-5x}{\sqrt{x^2+2}} \approx \dfrac{-5x}{\sqrt{x^2}} \approx \dfrac{-5x}{|x|} = \dfrac{-5(-\cancel{RBN})}{+\cancel{RBN}(\text{same as above})} = 5$

A. 5 B. –5 C. 0 D. $-\infty$ E. ∞

$\dfrac{-5(-3)}{\sqrt{(-3)^2}} = \dfrac{-5\cdot -3}{\sqrt{9}} = \dfrac{-5\cdot -3}{3} = \dfrac{+5\cdot\cancel{3}}{\cancel{3}} = 5$

21. The function $G(x)=\begin{cases} x-3, & x>2 \\ -5, & x=2 \\ 3x-7, & x<2 \end{cases}$ is not continuous at $x=2$ because…

$G(+2)=-5$

$\lim_{x\to 2^-} G(2)=3\cdot 2-7=6-7=-1$

A. $G(2)$ is not defined

B. $\lim_{x\to 2} G(x)$ does not exist

C. $\lim_{x\to 2} G(x) \neq G(2)$

D. Only reasons B and C

E. All of the above reasons.

$\lim_{x\to 2^+} G(2)=2-3=-1$

$\lim_{x\to 2} G(x)=-1$, $G(2)=-5$

$\lim_{x\to 2} G(x) \neq G(2)$

24. If $f(x)=3x^2-5x$, then find $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\frac{dy}{dx}=F'(x)=\frac{\Delta y}{\Delta x}$

A. $3x-5$

B. $6x-5$

C. $6x$

D. 0

E. Does not exist

$F(x)=3x^2-5x$

$F(x+h)=3(x+h)^2-5(x+h)$

$=3(x^2+2xh+h^2)-5(x+h)$

$$\lim_{h\to 0}\frac{\overbrace{3(x^2+2xh+h^2)-5(x+h)}^{F(x+h)}-\overbrace{[3x^2-5x]}^{F(x)}}{h}$$

$$\lim_{h\to 0}\frac{\cancel{3x^2}+6xh+3h^2-\cancel{5x}-5h-\cancel{3x^2}+\cancel{5x}}{h}=\lim_{h\to 0}\frac{\cancel{h}(6x+3h-5)}{\cancel{h}}$$

$6x+3(0)-5=6x-5$

7. $\lim_{x\to 0}\frac{3\tan x}{x\sec x}=\frac{\frac{3\sin x}{\cos x}}{\frac{x}{\cos x}}=\lim_{x\to 0}\frac{3\sin x}{\cancel{\cos x}}\cdot\frac{\cancel{\cos x}}{x}=\lim_{x\to 0}\frac{3\sin x}{x}=3\cdot 1=3$

$\cos(0)=1$

23. $\lim\limits_{x\to -2} \dfrac{\sqrt{2x+5}-1}{x+2} =$

$2x+4=2(x+2)$

A. 1 B. 0 C. ∞ D. $-\infty$ E. Does Not Exist

$$\lim_{x\to -2} \frac{(\sqrt{2x+5}-1)}{(x+2)} \cdot \frac{(\sqrt{2x+5}+1)}{(\sqrt{2x+5}+1)} = \lim_{x\to -2} \frac{2x+5-\cancel{\sqrt{2x+5}}+\cancel{\sqrt{2x+5}}-1}{(x+2)(\sqrt{2x+5}+1)}$$

$$\lim_{x\to -2} \frac{2x+5-1}{(x+2)(\sqrt{2x+5}+1)} = \lim_{x\to -2} \frac{2\cancel{(x+2)}}{\cancel{(x+2)}(\sqrt{2x+5}+1)} = \frac{2}{\sqrt{-4+5}+1} = \frac{2}{2}$$

26. The function $f(x) = \dfrac{2x^2+x-3}{x^2+4x-5}$ has a vertical asymptote at $x=-5$ because…

A. $\lim\limits_{x\to -5^+} f(x) = \infty = -\infty$ False

B. $\lim\limits_{x\to -5^-} f(x) = -\infty = +\infty$ False

C. $\lim\limits_{x\to -5^-} f(x) = \infty = +\infty$ True

D. ~~$\lim\limits_{x\to \infty} f(x) = -5$~~

E. $f(x)$ does not have a vertical asymptote at $x=-5$

$x^2+4x-5=(x+5)(x-1)$

$2x^2+x-3 \Rightarrow 2x^2+3x-2x-3$

$2\cdot 3=-6$

$3-2=1$

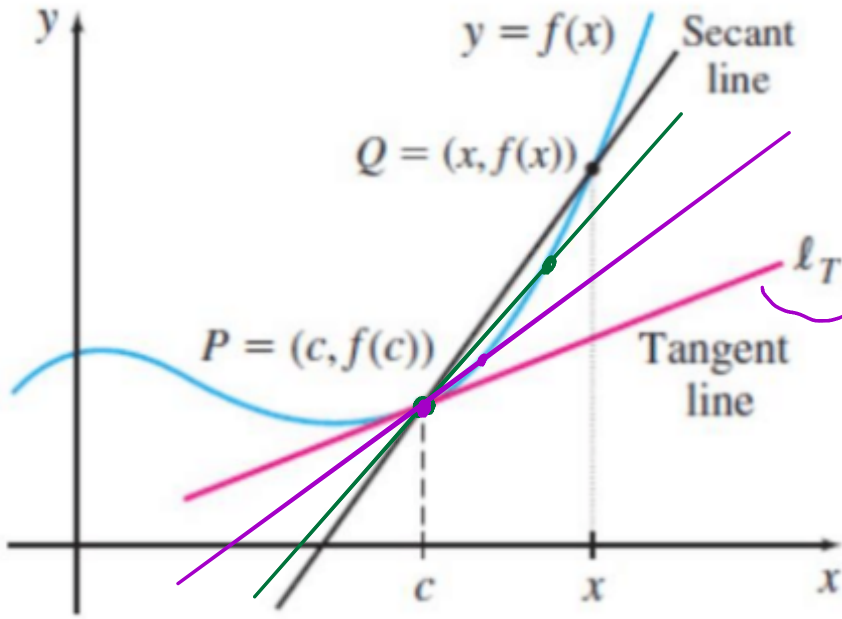
$x(2x+3)-1(2x+3)$

$x=1$ hole

$$\lim_{x\to -5} \frac{(2x+3)\cancel{(x-1)}}{(x+5)\cancel{(x-1)}}$$

$$\lim_{x\to -5^+} \frac{(2x+3)}{x+5} = \frac{-7}{+RSN} = -\infty \qquad \lim_{x\to -5^-} \frac{(2x+3)}{x+5} = \frac{-7}{-RSN} = +\infty$$

$x=-4.999$ $\qquad$ $x=-5.00001$



$$m = \lim_{h \to 0} \frac{F(x+h) - F(x)}{(x+h) - x}$$

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

Example 4:

Find the average rate of change of the function $f(x) = x^2 - 4$ on the interval [1,3]:

$F(3) = 3^2 - 4 = 5$

$F(1) = 1^2 - 4 = -3$

$$\frac{F(3) - F(1)}{3 - 1} = \frac{5 - (-3)}{3 - 1} = \frac{8}{2} = 4$$

$(3,5)(1,-3)$

$$IRC = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

$f(x) = x^2 - 4,$

$F(x+h) = (x+h)^2 - 4 = x^2 + 2xh + h^2 - 4$

$$\lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - 4 - [x^2 - 4]}{h}$$

$$\lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 \cancel{-4} \cancel{-x^2} \cancel{+4}}{h}$$

$$\lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} = 2x + 0 = 2x$$

$F'(x) = 2x$

$F'(2) = 2 \cdot 2 = 4$

$f\ f(x) = \frac{1}{x}$. Find $f'(1)$.

$F(x+h) = \frac{1}{x+h}$

$$\lim_{h\to 0} \frac{\frac{1}{(x+h)} \cdot \frac{x}{x} - \frac{1(x+h)}{x(x+h)}}{h} = \lim_{h\to 0} \frac{\frac{\cancel{x} \cancel{-x} - h}{x(x+h)}}{h} = \lim_{h\to 0} \frac{\frac{-h}{x(x+h)}}{\frac{h}{1}}$$

$$\lim_{h\to 0} \frac{-\cancel{h}}{x(x+h)} \cdot \frac{1}{\cancel{h}} = \frac{-1}{x^2} = F'(x) = \frac{dy}{dx} = \frac{\Delta y}{\Delta x}$$